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Critical behavior of branching annihilating random walks with an odd number of offsprings

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Recently, Takayasu and Tretyakov [Phys. Rev. Lett. **68**, 3060 (1992)] studied branching annihilating random walks with $n=1$—5 offsprings. These models exhibit a continuous phase transition to an absorbing state. Steady-state simulations yielded an estimate for the order parameter critical exponent β different from that of directed percolation. This result is quite surprising, as the universality class of directed percolation is known to be very robust. I have studied the critical behavior of the one-dimensional model with $n=1$ and 3 using time-dependent Monte Carlo simulations, and determined three critical exponents, all of which are in agreement with directed percolation.

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The study of nonequilibrium many-particle systems is an important problem in many branches of physics, chemistry, biology, and even sociology [1,2]. Special attention has been devoted to models exhibiting continuous phase transitions and in particular the question of determining the different universality classes. A common feature of many models is that they contain a single component, which we may think of as particles, and evolve according to a Markov process governed by local, intrinsically irreversible transition rules; such models are collectively known as *interacting particle systems* [3,4]. Examples are the contact process [5], a model for the spread of an epidemic, Schlögl's first and second models [6–9] for autocatalytic chemical reactions, surface reaction models [10,11], directed percolation (DP) in $(d+1)$ dimensions [12–14], and Reggeon field theory [15]. The evolution rules for these models typically involve spontaneous annihilation of particles, autocatalytic creation depending on the number of occupied neighbors, and possibly diffusion of particles on the lattice. As there is no spontaneous creation process the state with all sites vacant is an absorbing state for the Markov process. These models exhibit a (usually continuous) phase transition from an active steady state to the (unique) absorbing state when some external control parameter p exceeds a critical value p_c. The appropriate order parameter is normally just the concentration ρ of particles. The behavior of the order parameter in the vicinity of p_c may be described by a critical exponent β, $\rho \propto |p_c - p|^{\beta}$. One of the major achievements in the study of this type of model is the discovery that the models mentioned above have the same critical behavior, i.e., they belong to the same universality class. Studies of related models via computer simulations [16–19], field-theoretic arguments [9,20], and series expansions [21–23] demonstrate the robustness of this universality class against a wide range of changes in the local kinetic rules such as multiparticle processes, diffusion, and changes in the number of components. So at present there is substantial evidence in favor of the hypothesis [8,9,20], which I will refer to as the DP conjecture, that directed percolation or Reggeon field theory is the generic critical behavior of models with a scalar order parameter exhibiting a continuous transition to a unique absorbing state.

During the years several models have been reported to violate the DP conjecture. Originally it was reported that a one-dimensional cellular automaton proposed by Bidaux, Boccara, and Chaté belonged to a new universality class [24], as steady-state simulations yielded an estimate for β inconsistent with DP behavior. A study of this model using time-dependent computer simulations revealed that the model has DP-like behavior [25]. Likewise it was not clear for some time whether or not the Ziff-Gulari-Barshad model [10] for the oxidation of carbon monoxide on a catalytic surface was in the DP universality class, as steady-state simulations [26] yielded a value of β suggesting a different critical behavior. This question was finally settled in favor of DP behavior via time-dependent computer simulations [11] and field-theoretic renormalization-group arguments [20]. All this just goes to show that steady-state simulations can be tricky and that claims of non-DP critical behavior in this type of model should be taken with a grain of salt.

Another candidate for a violation of the DP conjecture is the branching annihilating random walk (BAW) introduced by Bramson and Gray [27]. In the BAW a particle is chosen at random. With probability p it jumps to a randomly chosen nearest neighbor, and if this site is already occupied both particles are annihilated. With probability $1-p$ the particle produces n offsprings, which are placed on the closest neighboring sites; if there is more than one possibility for placing the offsprings, a random choice is made. When an offspring is created on a site that is already occupied it annihilates with the occupying particle, leaving an empty site. In one dimension for $n=1$ it has been shown [27] that the BAW has an active steady state for sufficiently small p. Computer simulations revealed that the phase transition from the active state to the absorbing state is continuous [28]. As mentioned in the introduction, numerous models, including models with multiparticle annihilation or creation rules and diffusion [16,17,18,24], belong to the DP universality class. BAW's with an odd number of offsprings include a combination of such rules. One would therefore expect, bearing in mind the robustness of the DP critical behavior, that the transition should belong to the universality class of directed percolation. Results from computer simulations showed, however, that the exponent β differs from the directed percolation value. Takayasu and Tretyakov [28] found $p_c=0.108\pm0.001$ and $\beta=0.32\pm0.01$, which should be compared to the value $\beta^{\mathrm{DP}}=0.277\pm0.001$ [21] for directed percolation in $1+1$ dimensions. For $n=3$ and 5 they found that $p_c=0.461\pm0.002$ and 0.718 ± 0.001, respectively, with $\beta=0.33\pm0.01$ in both cases. These results signal a rather surprising violation of the DP conjecture. For $n=2$ the model does not have an active state [29], whereas for $n=4$ it was found that $\beta=0.7(1)$. The even n case is very special, as the number of particles is conserved modulo 2; this means that the absorbing state only exists if we start out with an even number of particles. If we start with an odd number of particles the system can never reach the absorbing state. One might then argue that the even n case should not be seen as violating the DP conjecture. A model very similar to the BAW with $n=2$ has been studied [30] via steady-state and time-dependent simulations, yielding non-DP values for various critical exponents.

Determining critical behavior from steady-state simulations is often very difficult due to large fluctuations, critical slowing down, finite-size effects, and difficulties in locating the critical point. In addition, special care has to be taken with simulations of a transition to an absorbing state, as one is studying a long-lived, but intrinsically metastable state. In this paper I present results from computer simulations of the one-dimensional BAW with $n=1$ and 3, using an alternative approach to the determination of critical exponents. The method used is known as time-dependent simulations. Earlier studies [7,11,14,17,18,25,31] have revealed that this is a very efficient method for determining critical points and exponents for models with a continuous transition to an absorbing state. The general idea of time-dependent simulations is to start from a configuration that is very close to the absorbing state, and then follow the "average" time evolution of this configuration by simulating a large ensemble of independent realizations. In the simulations presented here I always started, at $t=0$, with two occupied nearest-neighbor sites placed at the central sites of the lattice, and then made a number of independent runs, typically 1×10^5, for different values of p in the vicinity of p_c. As the number of particles is very small an efficient algorithm may be devised by keeping a list of occupied sites. In each elementary step a particle is drawn at random from this list and the processes are performed according to the rules given earlier. Before each elementary change the time variable is incremented by $1/n(t)$, where $n(t)$ is the number of particles on the lattices at that time. This makes one time step equal to (on the average) one attempted update per lattice site. Each run had a maximal duration of 2000 time steps. I measured the survival probability $P(t)$ (the probability that the system had not entered the absorbing state at time t), the average number of occupied sites $\bar{n}(t)$, and the average mean-square distance of spreading $\bar{R}^2(t)$ from the center of the lattice. Notice that $\bar{n}(t)$ is averaged over all runs, whereas $\bar{R}^2(t)$ is averaged only over the surviving runs. From the scaling ansatz for the contact process and similar models [7,14] it follows that the quantities defined above are governed by power laws at p_c as $t\to\infty$,

$$P(t)\propto t^{-\delta}\,, \tag{1}$$

$$\bar{n}(t)\propto t^{\eta}\,, \tag{2}$$

$$\bar{R}^2(t)\propto t^{z}\,. \tag{3}$$

In log-log plots of $P(t)$, $\bar{n}(t)$, and $\bar{R}^2(t)$ versus t we should asymptotically see a straight line at $p=p_c$. The curves will show positive (negative) curvature when $p>p_c$ ($p<p_c$). This makes it possible to obtain accurate estimates for p_c. The asymptotic slopes of the (critical) curves define the dynamic critical exponents δ, η, and z. Generally one has to expect corrections to the pure power-law behavior, so that $P(t)$ is more accurately given as [14]

$$P(t)\propto t^{-\delta}(1+at^{-1}+bt^{-\delta'}+\cdots)\,, \tag{4}$$

and similarly for $\bar{n}(t)$ and $\bar{R}^2(t)$. More precise estimates for the critical exponents can be obtained if one looks at local slopes

$$-\delta(t)=\frac{\ln[P(t)/P(t/m)]}{\ln(m)}\,, \tag{5}$$

and similarly for $\eta(t)$ and $z(t)$; in this work I used $m=5$. The local slope $\delta(t)$ behaves as [14]

$$\delta(t)=\delta+at^{-1}+b\delta' t^{-\delta'}+\cdots\,, \tag{6}$$

and similar expressions for $\eta(t)$ and $z(t)$. Thus in a plot of the local slopes versus $1/t$ the critical exponents are given by the intercept of the curve for p_c with the y axis. The off-critical curves often have a very notable curvature, i.e., one will see the curves for $p < p_c$ veering downward, while the curves for $p > p_c$ veer upward.

Figure 1 shows the results for $n = 1$ and Fig. 2 the results for $n = 3$. From the results, in particular, the curves for $\eta(t)$, I estimate that $p_c = 0.1070 \pm 0.0005$ for the BAW with $n = 1$ and $p_c = 0.59 \pm 0.001$ for the BAW with $n = 3$. These estimates for p_c are consistent with the results obtained by Takayasu and Tretyakov [28] within the cited uncertainty, although the results of the present work generally favor a lower center value. From the curves closest to p_c I obtain the following estimates for the critical exponents: $\delta = 0.160 \pm 0.005$, $\eta = 0.31 \pm 0.01$, and $z = 1.26 \pm 0.01$ for $n = 1$; $\delta = 0.165 \pm 0.005$, $\eta = 0.305 \pm 0.010$, and $z = 1.26 \pm 0.01$ for $n = 3$.

The values for δ, η, and z are in excellent agreement with the values for a lattice version of Reggeon field theory (RFT) in one dimension as obtained by simulations [7]: $\delta = 0.162 \pm 0.004$, $\eta = 0.308 \pm 0.009$, and $z = 1.263 \pm 0.008$. Series expansions for RFT [15] yielded $\delta = 0.160 \pm 0.001$ [32], $\eta = 0.317 \pm 0.002$, and $z = 1.272 \pm 0.007$, which again is in very good agreement with the simulation results presented in this article. As a further

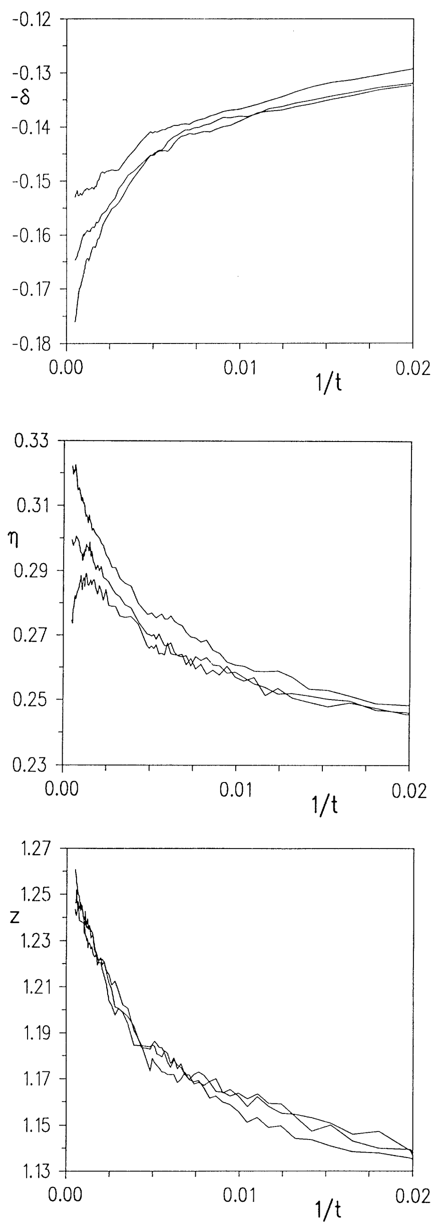

FIG. 1. The local slopes $-\delta(t)$ (upper panel), $\eta(t)$ (middle panel), and $z(t)$ (bottom panel), for the BAW with $n = 1$. Each panel contains five curves with from bottom to top $p = 0.1060$, 0.1065, 0.1070, 0.1075, and 0.1080.

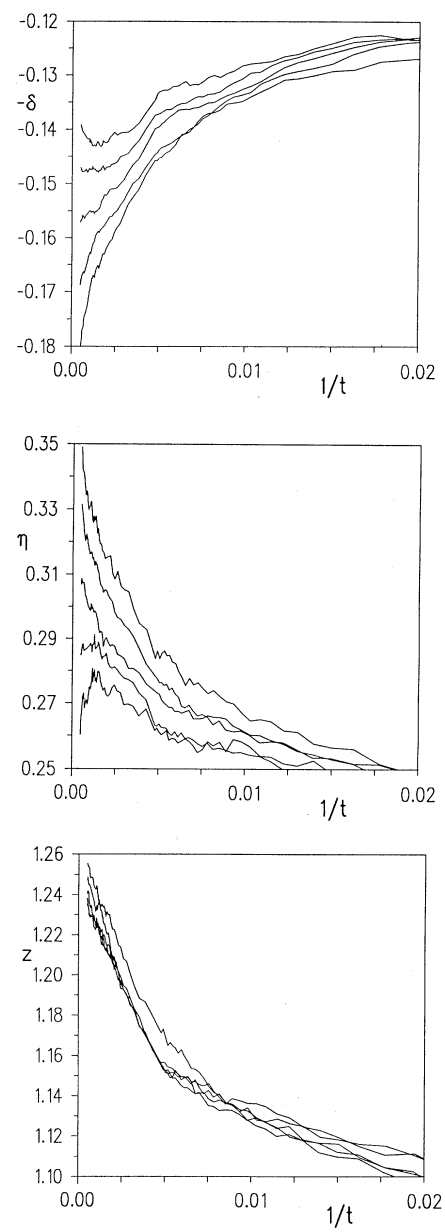

FIG. 2. The local slopes $-\delta(t)$ (upper panel), $\eta(t)$ (middle panel), and $z(t)$ (bottom panel), for the BAW with $n = 3$. Each panel contains three curves with from bottom to top $p = 0.458$, 0.459, and 0.460.

test of the consistency of the data I have used a well-established [7] scaling relation between the exponents:

$$\delta=\frac{1}{2}\left[\frac{z}{2}d-\eta\right]. \tag{7}$$

It is clearly seen that the estimates for the BAW given above agree very well with this scaling relation.

Takayasu and Tretyakov obtained a static critical exponent, and it may be possible that the static exponents are non-DP, though the dynamic exponents agree with DP. Note, however, that the scaling relation [7] $\beta=\delta/\nu$, where ν is the correlation length exponent in the time direction, links the static and dynamic exponents. The arguments of Ref. [7] are not rigorous but even so do indicate that this possibility is small. Another major difference between BAW's and DP is that the critical dimension of the DP is known to be 4, whereas the results of Takayasu and Tretyakov indicate that the critical dimension for BAW's is 2. BAW's do not require a specific neighboring configuration, e.g., in the BAW with $n=3$ a "creation" event always takes place with probability $1-p$, even though all three neighbors are not empty. Effectively a creation event can lead to the creation of three or one new particle or the destruction of one or three particles, depending on the number of occupied neighbors. This contrasts with most other models in which a multiparticle creation event would only happen if the required number of nearest neighbors were all empty. This insensitivity to the details of the environment might explain why the critical dimension of the BAW seems to be 2 and thus lower than the critical dimension of DP, which is known to be 4. This situation is somewhat similar to the Bidaux-Boccara-Chaté (BBC) model [24] in which the evolution rules are also insensitive to the detailed particle configuration. The one-dimensional BBC model exhibits a continuous phase transition belonging to the DP universality class [25]. In higher dimensions the BBC model exhibits a *first-order* phase transition, in agreement with the predictions of mean-field theory [24].

All in all I conclude that the BAW with $n=1$ and 3 offsprings very likely belongs to the universality class of directed percolation and thus does not constitute a violation of the DP conjecture. Given the stability of this universality class, which is further supported by this work, one would expect that all BAW's with an odd number of offsprings belong to the DP universality class. The non-DP estimates for β reported by Takayasu and Tretyakov may be caused by the slight misplacement in their estimate for p_c as compared to this work.

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